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Quantum Group Symmetry of the Hubbard Model

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Quantum group symmetry is shown to exist for the Hubbard model. It is extended to include infinitesimally deformed phonons. Also a simplified version of Alam's model is generalized to include phonons and is shown to have quantum group symmetry.

The Hubbard model (Hubbard, 1963) has been much studied, since it models strongly interacting electron systems. Sometimes it is also used to study high-temperature superconductivity (HTSC). Therefore studying its symmetries and extending them (if possible) is an important task. Several authors (Heilmann and Lieb, 1970; Yang and Zhang, 1990) have shown that it has a $SO(4) = SU(2) \times SU(2)/Z_2$ symmetry at half-filling. Later it was shown that it has a quantum symmetry, $SU_q(2) \times SU_q(2)/Z_2$ (Montrosi and Rasetti, 1994), and was is argued (Schupp, 1997) that this symmetry can be obtained from ordinary symmetry via "twists," which are not equivalence transformations. We prefer to work directly with the quantum group symmetry for reasons that will be mentioned later. Recently it has been argued that quantum group symmetry is related to HTSC (Alam and Rahman, 1999). Here the model of Montrosi and Rasetti is extended to include infinitesimally deformed (Ahmed and Hegazi, 1990) phonons so both fermionic and bosonic sectors of the model are deformed. Also a simplified version the stripes model of Alam and Rahman is extended to include phonons so that its quantum group symmetry becomes explicit.

We begin with the standard Hubbard Hamiltonian H_1,

$$H_1 = u \sum_i n_{i\uparrow} n_{i\downarrow} - \mu \sum_{i,\sigma} n_{i\sigma} + t \sum_{\langle ij\rangle,\sigma} a_{j\sigma}^\dagger a_{i\sigma}, \qquad n_{i\sigma} = a_{i\sigma}^\dagger a_{i\sigma} \tag{1}$$

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where $a_{i\sigma}$, $a^{\dagger}_{i\sigma}$ are the fermionic operators with spin $\sigma \in \{;\uparrow, \downarrow\}$ at site i, and $\langle i, j\rangle$ are nearest neighbor distinct sites. The generators of $SU(2) \times SU(2)/Z_2$ symmetry are

$$J^{+}_{m} = a^{\dagger}_{\uparrow}a_{\downarrow}, \qquad J^{-}_{m} = (J^{+}_{m})^{\dagger}, \qquad J^{3}_{m} = n_{\uparrow} - n_{\downarrow} \tag{2}$$

$$J^{+}_{s} = a^{\dagger}_{\uparrow}a^{\dagger}_{\downarrow,} \quad J^{-}_{s} = (J^{-}_{s})^{\dagger}, \qquad J^{3}_{s} = n_{\uparrow} + n_{\downarrow} - 1 \tag{3}$$

The Z_2 symmetry comes from the conjugation $a_{\downarrow} \leftrightarrow a^{\dagger}_{\downarrow}$. The invariance of the first two terms in (1) requires half filling i.e. $\mu = u/2$. To include the nonlocal (third) term, the global operators [e.g., $J^{+}_{m} = \Sigma_i\, a^{\dagger}_{i\uparrow}a_{i\downarrow}$ and similarly for the other operators in (2) and (3)] are used.

The generators in (2) and (3) satisfy the $SU_q(2)$ relations

$$[J^{+}_{s}, J^{-}_{s}] = (q^{J^{3}_{s}} - q^{-J^{3}_{s}})/(q - q^{-1}), \qquad [J^{3}_{s}, J^{\pm}_{s}] = \pm 2J^{\pm}_{s} \tag{4}$$

and similarly for the magnetic operators. The proof uses $(J^3)^3 = J^3$. The coproduct of the quantum group is defined by

$$\Delta_q(J^{\pm}) = J^{\pm} \otimes q^{-J^{3}/2} + q^{J^{3}/2} \otimes J^{\pm}, \qquad \Delta_q(J_3) = 1 \otimes J^3 + J^3 \otimes 1$$

To include the nonlocal term in the quantum symmetry the Hamiltonian H_1 in (1) can be modified to include phonons (Montrosi and Rosetti, 1994)

$$H_2 = u \sum_i n_{i\uparrow}n_{i\downarrow} - \mu \sum_{i,\sigma} n_{i\sigma} - \boldsymbol{\lambda} \cdot \sum_{i,\sigma} n_{i\sigma}\mathbf{x}_i + \sum_i [(\mathbf{p}^2)_i/2m + \tfrac{1}{2}m\omega^2\mathbf{x}^2_i] + \sum_{\langle i,j\rangle,\sigma} T_{ij}a^{\dagger}_{j,\sigma}a_{j\sigma} + \text{h.c.} \tag{5}$$

where $\mathbf{x}_i$ is the displacement of the ions, $\mathbf{p}_i$ is the corresponding momentum, and

$$T_{ij} = t\,\exp[i\boldsymbol{\kappa} \cdot (\mathbf{p}_i - \mathbf{p}_j)], \qquad J^{\pm} = J^{\pm}\,\exp[\mp 2i\boldsymbol{\lambda} \cdot \mathbf{p}/\hbar m\omega^2] \tag{6}$$

The Hamiltonian H_2 is invariant under $SU_q(2) \otimes SU_q(2)/Z_2$ provided that

$$\mu = u/2 - \boldsymbol{\lambda}^2/m\omega^2, \qquad \boldsymbol{\lambda} = \hbar m\omega^2\boldsymbol{\kappa} \tag{7}$$

This symmetry is expected to be valid if $d = 1$.

So far the quantum symmetry is for the fermionic sector; therefore it is proposed here to used infinitesimally deformed (Ahmed and Hegazi, 1990) phonons instead of ordinary ones. They are defined by the relations

$$[x, p] = i\hbar + i\varepsilon H_0, \qquad [H, p] = \tfrac{1}{2}\hbar m\omega^2 x, \qquad [H, x] = -p\hbar/2m$$
$$H = H_0 - \varepsilon H^2_0, \qquad H_0 = \mathbf{p}^2/2m + m\omega^2\,\mathbf{x}^2/2 \tag{8}$$

The bosonic deformation parameter is $q_b \simeq 1 + \varepsilon$ and ones linearize in ε.

The new Hamiltonian is $H_3 = H_2 - \varepsilon \Sigma_i (\mathbf{p}_i^2/2\mathrm{m} + \frac{1}{2}m\omega^2\mathbf{x}_i^2)^2$ and the symmetry requires

$$\boldsymbol{\lambda} = \hbar m\omega^2\boldsymbol{\kappa}, \qquad \mu = u/2 - \boldsymbol{\lambda}^2/m\omega^2[1 + \varepsilon \Sigma_i (1/2m\mathbf{p}_i^2 + \tfrac{1}{2}m\omega^2\mathbf{x}_i^2)] \tag{9}$$

$$\begin{aligned} H_3 = {} & H_2(a_{i\sigma}, a_{i\sigma}^\dagger, u, \mu, m, \omega, \boldsymbol{\lambda}, p_i, \mathbf{x}_i, \boldsymbol{\kappa}) \\ & + H_2(b_{i\sigma}, b_{i\sigma}^\dagger, u', \mu', m, \omega, \boldsymbol{\lambda}, \mathbf{p}_i, \mathbf{x}_i, \boldsymbol{\kappa}) \\ & + \zeta \sum_{i,\sigma,\tau} n_{i\sigma}n'_{i\tau} \qquad (\text{where } n'_{i\sigma} = b_{i\sigma}^\dagger\, b_{i\sigma}) \end{aligned} \tag{10}$$

The quantum symmetry requires

$$\mu = u/2 - \boldsymbol{\lambda}^2/m\omega^2, \qquad \mu' = u'/2 - \boldsymbol{\lambda}^2/m\omega^2, \qquad \boldsymbol{\lambda} = hm\omega^2\,\boldsymbol{\kappa} \tag{11}$$

There are two interesting features of the quantum symmetry of the Hubbard model. The first is that phonons can be included naturally without breaking the symmetry. This is expected to be relevant since some HTSC materials show a nonzero isotope effect, which indicates that phonons should not be negligible. The second feature is that both magnetic and superconducting symmetries are included. Again this is expected to be relevant to HTSC. These are strong motivations that quantum groups are relevant to HTSC, at least as a starting point. It is hoped that this work, together with others, is a step in the right direction.

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